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Refinement on the Descriptions of b-Bistochastic Quadratic Stochastic Operators and their Fixed Points' Stability

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ABSTRACT

Quadratic Stochastic Operators (QSOs) was originally used to solve a problem relating to genetics by Bernstein in 1942. Since then, QSOs had seen applications in various fields, that is why subclasses of QSOs were introduced to fit their respective niches. In this paper, we will consider b-bistochastic QSOs. Previous theorem on the descriptions of b-bistochastic QSOs on two-dimensional simplex was refined and the full description of b-bistochastic QSOs on any finite dimensional simplex are given. One of the main problems in nonlinear operator theory is to study the dynamics and the omega limiting set of the associate operators. Since the fixed points are the subset of the limiting set, thus it is nature to list down all the fixed points. Here, we limit ourselves to three-dimensional simplex. Next, it is shown that this operator does not have any interior fixed point, which exhibits the same properties as in lower dimensions. The behaviour of such fixed points was investigated in numerically. This understanding will be the stepping stone for the dynamics of b-bistochastic QSOs on three-dimensional simplex and higher dimensional simplexes as well.

Keywords: Quadratic Stochastic Operators, doubly stochastic, fixed points

INTRODUCTION

The topic on Quadratic Stochastic Operators (QSOs) may be traced back to the early origins of Bernstein (1942). While describing the mathematical description of a heredity problem, Bernstein discovered that a QSO fits the necessary description of the problem. QSOs are widely used in genetics and heredity after Bernstein's groundbreaking work.

Due to its usefulness and very simple structure, QSOs may be found in a plethora of several other topics. In biology, one may use QSOs to model interactions of species with non-overlapping generations (Hofbauer et al., 1987). In ecology, the complex dynamics of simple ecological models may be described with QSOs (May and Oster, 1976). In physics, Lotka-Volterra QSOs were investigated as a type of Hamiltonian system (Plank, 1995).

On the topic of genetics, we can see how some species may be modelled as a QSO. If we have n types of species (or traits) in a population with initial probability distribution being $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}\right)$ as well as $P_{ij,k}$ being the heredity coefficient, the probability of obtaining some kth species (traits) by cross-fertilisation between ith and jth species (traits) where $1 \leq i,j,k \leq n$. Then the first generation of species (traits), $\boldsymbol{x}^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}\right)$ is determined by the total probability, forming the QSO. That is

$$\boldsymbol{x}^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = 1,2,\dots,n.$$

The above operator can be simplified as the operator V . The interesting thing about this operator is that, given some arbitrary initial condition $\boldsymbol{x}^{(0)}$, it is possible to continuously evolve the population to get the first generation, $\boldsymbol{x}^{(1)} = V(\boldsymbol{x}^{(0)})$, second generation $\boldsymbol{x}^{(2)} = V(\boldsymbol{x}^{(1)}) = V\left(V(\boldsymbol{x}^{(0)})\right) = V^2(\boldsymbol{x}^{(0)})$, and so on. Thus, the population may be described by the following dynamical system:

$$\boldsymbol{x}^{(0)}, \quad \boldsymbol{x}^{(1)} = V(\boldsymbol{x}^{(0)}), \quad \boldsymbol{x}^{(2)} = V^2(\boldsymbol{x}^{(0)}), \quad \boldsymbol{x}^{(3)} = V^3(\boldsymbol{x}^{(0)}), \quad \dots$$

Many applications of QSOs have come about through dynamical systems. One of the most common uses of QSOs are in population dynamics. Rozikov and Shoyimardonov (2019) applied l-Volterra QSOs on modelling the concentration of nutrients, phytoplanktons and zooplanktons in an NPZ model, specifically on the limiting points. Rozikov and Usmonov (2020) investigated the dynamics and the limiting points a population with two equal dominated species, in which the model is modelled as a QSO. Other than in population dynamics, QSOs has also seen applications in game theory. Ganikhodjaev et al. (2015) integrated QSOs into game theory in investigating a zero-sum game, namely rock-paper-scissors. It is discovered that an extremal Volterra QSO on S^4 was able to describe the trajectory behaviour of the game.

Various types of QSOs had also seen research up to recent years. Lim (2024) published a paper on developing a C++ program that solves Geometric QSOs. Qaisar (2025) investigated the dynamics of $\xi^{(\alpha)}$ QSOs and built Quadratic Stochastic Processes (QSP) that is modeled from the Susceptible-Infected-Recovered (SIR) model. Ganikhodjaev et al. (2024) investigated some properties of fixed points of Volterra QSOs on an infinite-dimensional simplex and had shown it is non-regular. Khaled and Pah (2021) introduced a new form of QSO known as mixing geometric QSOs which all QSO generated by such geometric distributions is regular. Eshmamatova et al. (2025) constructed the fixed-point cards for Volterra QSOs on a four-dimensional simplex.

On the other hand, concept of majorization was established by Hardy et al. (1952). The concept of majorization acts as a method to see how "spread apart" two sequences of numbers are to each other. This concept has seen many applications in multiple fields especially those in economy (Kleiner et al., 2021) as it is helpful in determining income gap and wealth inequality. It may also be used in statistics, an example of a problem on a 'minimal repair' (Boland and El-Neweihi, 1998) applies the concepts of majorization.

Parker and Ram (1996) introduced a new order called majorization built on Hardy's concept of majorization, in which they refer to as classical majorization. The advantage of this majorization over classical majorization is that majorization can be defined as a partial ordering on sequences whereas classical majorization is only defined as a preorder on sequences. However, Parker and

Ram focused on majorization defined as a linear system only. Therefore, it is in the interest of this paper in investigating non-linear cases, specifically on quadratic systems.

In 1993, Ganikhodzhaev provided a definition of bistochastic QSOs in terms of classical majorization (Ganikhodzhaev, 1993a). In this case, a QSO is said to be bistochastic (or doubly stochastic) if $V(\boldsymbol{x}) \prec \boldsymbol{x}$ for all $\boldsymbol{x}$ in the $n-1$ dimensional simplex where $\prec$ refers to classical majorization.

Bistochastic QSOs had seen many applications especially on consensus problems. Saburov and Saburov (2024) highlighted the applications of QSOs on nonlinear consensus problems in which bistochastic QSOs were considered. Abdulghafor (2024) also used bistochastic QSOs as a model for investigating nonlinear consensus for multi-agent systems (MAS).

Having seen the application of classical majorization into the QSOs, the logical next step is to consider whether Parker and Ram's majorization may do the same. In order to differentiate the classical majorization and Parker and Ram's majorization, from this point onwards Parker and Ram's majorization will be referred to as b-order whereas classical majorization will be referred to as majorization. In 2015, Mukhamedov and Embong (2015) defined a new class of QSOs by applying the concept of b-ordering. This new class of QSO was named as b-bistochastic QSOs.

In the same paper by Mukhamedov and Embong (2015), a description of b-bistochastic QSOs for two-dimensional simplexes had been provided. In 2023, Embong and Rosli (2023) had researched the fixed point and dynamics of two-dimensional b-bistochastic QSOs. It can be seen that the description, fixed points and dynamics of b-bistochastic QSOs has only been investigated up to two-dimensional simplexes. The fixed points of b-bistochastic QSOs and its dynamics investigated by Embong and Rosli (2023). Embong et al. (2024) provided a full description for the canonical form for a class of two-sex population namely b-bistochastic Volterra QSO. Therefore, this paper aims to extend and to refine this understanding to three-dimensional simplexes. By this extension, it is possible to obtain new insight on these behaviours on any finite dimensional simplexes.

PRELIMINARIES AND BASIC DEFINITIONS

Let S^{n-1} denote the $(n-1)$-dimensional simplex in which

$$S^{n-1} = \boldsymbol{x} = \left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \,|x_i \geq 0 \; and \; \sum_{i=1}^{n} x_i = 1\right\}. \tag{1}$$

Let V be an operator that maps to itself, $V: S^{n-1} \to S^{n-1}$ with the form

$$V(\boldsymbol{x})_k = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j\,, \;\; \boldsymbol{x} = (x_1, x_2, \dots, x_n) \in S^{n-1} \;\; where \;\; k = 1,2,\dots,n, \tag{2}$$

where

$$P_{ij,k} \geq 0, \qquad P_{ij,k} = P_{ji,k}, \qquad \sum_{k=1}^{n} P_{ij,k} = 1, \qquad i,j,k = 1,2,\dots,n. \tag{3}$$

Let $\mathcal{U}_k: \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathcal{U}_k(x_1, x_2, \dots, x_n) = \sum_{i=1}^{k} x_i \quad where\ k = 1,2, \dots, n-1. \tag{4}$$

Let $x, y \in S^{n-1}$, define $\boldsymbol{x}$ to be *b-ordered* by $\boldsymbol{y}$ ($x \leq^b y$) if and only if $U_k(x) \leq U_k(y)$ for all $k = 1,2, \dots, n-1$.

Let $x \in S^{n-1}$. Define $x_{[\downarrow]} = (x_{[1]}, x_{[2]}, x_{[3]}, \dots, x_{[n]})$ where $x_{[1]} \geq x_{[2]} \geq x_{[3]} \geq \cdots \geq x_{[n]}$. $x_{[\downarrow]}$ is called the rearragement of x by nonincreasing. Take two elements $\boldsymbol{x}, \boldsymbol{y} \in S^{n-1}$, it is said that $\boldsymbol{x}$ is *majorized* by $\boldsymbol{y}$ ($\boldsymbol{x} \prec \boldsymbol{y}$) if $x_{[\downarrow]} \leq^b y_{[\downarrow]}$. Using the previous definitions, a definition *b*-bistochastic QSOs is as follows.

Definition 1. Let V be a QSO in the form of (2) and S^{n-1} be the $(n-1)$-dimensional simplex. V is a b-bistochastic QSO if

$$V(x) \leq^b x, \qquad \forall x \in S^{n-1}.$$

The following definition will be used for defining the fixed points.

Definition 2. Let $V: S^{n-1} \to S^{n-1}$, then $x = (x_1, x_2, \dots, x_n) \in S^{n-1}$ is a fixed point of V if $V(x)_i = x_i$ for $i = 1,2, \dots, n$. The list of all fixed points of V is denoted by $Fix(V)$.

DESCRIPTION OF *b*-BISTOCHASTIC QSO

The following lemma is necessary.

Lemma 1 (Mukhamedov and Embong (2015)). *The inequality*

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n + C \leq 0 \tag{5}$$

holds under the condition $0 \leq x_1 + x_2 + \cdots + x_n \leq 1$ *if and only if*

(1) $C \leq 0$ *and*

(2) $A_k + C \leq 0$ *for all* $k = 1,2, \dots, n$.

Next, we will provide the descriptions of b-bistochastic QSOs on one- and two-dimensional simplexes that was proven by Mukhamedov and Embong (2015).

Recall, the QSO on this simplex, $V: S^1 \to S^1$ is in the form

$$V(\boldsymbol{x})_1 = Ax_1^2 + 2Bx_1(1 - x_1) + C(1 - x_1)^2,$$

where $P_{11,1} = A, P_{12,1} = B$ and $P_{22,1} = C$.

Theorem 2 (Mukhamedov and Embong (2015)). *Let* $V: S^1 \to S^1$, V *is a b-bistochastic QSO if and only if*

$$C = 0, \qquad B \leq \frac{1}{2}.$$

For two-dimensional simplex, V takes the form of

$$V(\boldsymbol{x})_1 = A_1{x_1}^2 + 2B_1x_1x_2 + 2C_1x_1(1 - x_1 - x_2) + D_1{x_2}^2 + 2E_1x_2(1 - x_1 - x_2) + F_1(1 - x_1 - x_2)^2, \quad (6)$$

$$V(\boldsymbol{x})_2 = A_2{x_1}^2 + 2B_2x_1x_2 + 2C_2x_1(1 - x_1 - x_2) + D_2{x_2}^2 + 2E_2x_2(1 - x_1 - x_2) + F_2(1 - x_1 - x_2)^2, \quad (7)$$

where $x_1, x_2 \in [0,1]$ and
$A_1 = P_{11,1}, \quad C_1 = P_{13,1}, \quad E_1 = P_{23,1},$
$A_2 = P_{11,2}, \quad C_2 = P_{13,2}, \quad E_2 = P_{23,2},$
$B_1 = P_{12,1}, \quad D_1 = P_{22,1}, \quad F_1 = P_{33,1},$
$B_2 = P_{12,2}, \quad D_2 = P_{22,2}, \quad F_2 = P_{33,2}.$

We denote, $a = A_1 + A_2 + D_2 - 2B_1 - 2B_2,\ b = 2B_1 + 2B_2 - 2D_2,\ c = D_2 - 1.$

The following theorem describes b-bistochastic QSOs on two-dimensional simplex.

Theorem 3 (Mukhamedov and Embong (2015)). *Let $V: S^2 \to S^2$ be a QSO, then V is b-bistochastic if and only if*

(1) $F_1 = E_1 = D_1 = F_2 = 0$;
(2) $B_1 \leq \frac{1}{2},\ C_1 \leq \frac{1}{2},\ E_2 \leq \frac{1}{2}$;
(3) $C_1 + C_2 \leq \frac{1}{2}$;

with one of the following satisfied:

1. $a \geq 0$
2. $a < 0$ *and one of the following satisfied:*
    a. $b \geq 0$
    b. $b \geq -2a$
    c. $b^2 - 4ac \leq 0$

The necessary conditions for finite dimensional b-bisotchastic QSO is given by the following theorem.

Theorem 4 (Mukhamedov and Embong (2015)) *Let V be a b-bistochastic QSO defined on S^{n-1}, then the following properties hold:*

(i) $\sum_{m=1}^{k} \sum_{j=1}^{n} P_{ij,m} \leq kn;\ k = \underline{1, n}$

(ii) $P_{ij,k} = 0$ *whenever i and $j \geq k + 1$.*

(iii) $P_{nn,n} = 1.$

(iv) (i) $V(\boldsymbol{x})_k = \sum_{l=1}^{k} P_{ll,k} x_l^2 + 2\sum_{i=l}^{k}\sum_{j=l+1}^{n} P_{ij,k}\, x_i x_j$ *where* $k = 1,2,\dots,n-1$

(ii) $V(\boldsymbol{x})_n = x_n^2 + \sum_{l=1}^{n-1} P_{ll,n} x_l^2 + 2\sum_{l=1}^{n-1}\sum_{j=l+1}^{n} P_{lj,n}\, x_i x_j$

(v) $P_{lj,l} \le \frac{1}{2}$ *for all* $l = 1,2,\dots,n-1$ *where* $j \ge l+1$

(vi) $P_{ll,l} + 2\sum_{j=l+1}^{n} P_{lj,l} + 2P_{ln,l}(l-n) \le 1$ *where* $l = 1,2,\dots,n$.

However, the sufficient conditions were yet to be discovered. Therefore, we are able to improve the theorem by providing the full description of b-bistochastic QSOs on any finite dimensional simplex. The result is as stated in the theorem below.

Theorem 5. *Let* $V: S^{n-1} \to S^{n-1}$ *be a QSO,* V *is a b-bistochastic if and only if*

1. $P_{ij,k} = 0$ *whenever* i *and* $j \ge k+1$ *for any* $k = 1,2,\dots,n$,
2. $\sum_{l=i}^{k} P_{ij,l} \le \frac{1}{2}$ *whenever* $j \ge k+1$ *for any* $i = 1,2,\dots k$ *and* $k = 1,2,\dots,n$.

Proof. First, we assume that V is b-bistochastic, that is to say

$$\sum_{l=1}^{k} V(\boldsymbol{x})_l \le x_1 + x_2 + x_3 + \dots + x_k$$

for some fixed $k = 1,\dots,n$. It is clear that condition (i) can be obtained from Theorem 4(ii). By applying $P_{ij,k} = 0$ into (2), we obtain

$$\begin{aligned} V(\boldsymbol{x})_1 &= P_{11,1}x_1^2 + 2P_{12,1}x_1x_2 + 2P_{13,1}x_1x_3 + \dots + 2P_{1n,1}x_1(1 - x_1 - x_2 - \dots - x_{n-1}),\\ V(\boldsymbol{x})_2 &= P_{11,2}x_1^2 + 2P_{12,2}x_1x_2 + 2P_{13,2}x_1x_3 + \dots + 2P_{1n,2}x_1(1 - x_1 - x_2 - \dots - x_{n-1})\\ &\quad + P_{22,2}x_2^2 + 2P_{23,2}x_2x_3 + \dots + 2P_{2n,2}x_2(1 - x_1 - x_2 - \dots - x_{n-1}),\\ &\vdots\\ V(\boldsymbol{x})_l &= P_{11,n}x_1^2 + 2P_{12,n}x_1x_2 + 2P_{13,n}x_1x_3 + \dots + 2P_{1n,n}x_1(1 - x_1 - x_2 - \dots - x_{n-1})\\ &\quad + P_{22,n}x_2^2 + 2P_{23,n}x_2x_3 + \dots + 2P_{2n,n}x_2(1 - x_1 - x_2 - \dots - x_{n-1}) + \dots\\ &\quad + P_{ll,l}x_l^2 + 2P_{l(l+1),l}x_lx_{l+1} + \dots + 2P_{ln,l}x_l(1 - x_1 - x_2 - \dots - x_{n-1}). \end{aligned}$$

By collecting all like terms from each $V(\boldsymbol{x})_l$, one may obtain this form,

$$\sum_{l=1}^{k} V(x)_l = \sum_{i=1}^{k}\sum_{l=i}^{k} P_{ii,l}x_i^2 + 2\sum_{i=1}^{k}\sum_{j=i+1}^{n}\sum_{l=i}^{k} P_{ij,l}x_ix_j \le x_1 + x_2 + \dots + x_k. \tag{8}$$

Now, choose $x = (0,0,\dots,0,x_p,0,\dots,0,x_q,0,\dots,0) \in S^{n-1}$ such that $x_p, x_q > 0$ and $p < k+1 \le q$. It can be seen that (8) can be simplified to

$$x_p^2\sum_{l=p}^{k}P_{pp,l}+2\sum_{l=p}^{k}P_{pq,l}x_px_q \ \le\ x_p, x_p\left(x_p\sum_{l=p}^{k}P_{pp,l}+2\sum_{l=p}^{k}P_{pq,l}x_q-1\right)\ \le\ 0.$$

Taking into account $x_p > 0$ and $x_p + x_q = 1$ then,

$$x_p\sum_{l=p}^{k}P_{pp,l}+2\sum_{l=p}^{k}P_{pq,l}(1-x_p)-1\ \le\ 0,$$
$$x_p\sum_{l=p}^{k}(P_{pp,l}-P_{pq,l})+2\sum_{l=p}^{k}P_{pq,l}-1\ \le\ 0.$$

By applying Lemma 1, one finds that $2\sum_{l=p}^{k}P_{pq,l}-1\le 0$ which becomes $\sum_{l=p}^{k}P_{pq,l}\le\frac{1}{2}$. Since both p, q were chosen arbitrarily, the statement then generalises to

$$\sum_{l=i}^{k}P_{ij,l}\le\frac{1}{2},\quad whenever\ j\ge k+1\ for\ any\ i=1,2,\dots,k.$$

This concludes the first part of the proof. Now, we assume that (i) and (ii) are true and we aim to proof that V is b-bistochastic.
First, we take the form (8). By splitting up the summation at $j = k$ and $j = n$, one can obtain the following form.

$$\begin{aligned}\sum_{l=1}^{k}V(\boldsymbol{x})_l &= \sum_{i=1}^{k}\sum_{l=i}^{k}P_{ii,l}x_i^2+2\sum_{i=1}^{k}\sum_{j=i+1}^{n}\sum_{l=i}^{k}P_{ij,l}x_ix_j\\ &= \sum_{i=1}^{k}\sum_{l=i}^{k}P_{ii,l}x_i^2+2\sum_{i=1}^{k}\sum_{j=i+1}^{k}\sum_{l=i}^{k}P_{ij,l}x_ix_j+2\sum_{i=1}^{k}\sum_{j=k+1}^{n-1}\sum_{l=i}^{k}P_{ij,l}x_ix_j\\ &\quad+2\sum_{i=1}^{k}\sum_{l=i}^{k}P_{in,l}x_i(1-x_1-x_2-\cdots-x_{n-1})\end{aligned}$$

By properties of the heredity coefficient, $\sum_{k=1}^{n}P_{ij,k}\le 1$ and property (ii). It can also be noted that in the final summation of the equation, one finds that the terms where $i\ne j$ occurs two times when $i<j\le k$. For example, x_1x_2 occurs once when $i = 1$ and $i = 2$. When $i\le k<j$ occurs only once, since $P_{ij,k}=0$ whenever i and $j\ge k+1$ by property (i). Therefore, the equation then becomes the following inequality,

$$\begin{aligned}\sum_{l=1}^{k}V(\boldsymbol{x})_l = \sum_{i=1}^{k}\sum_{l=i}^{k}P_{ii,l}x_i^2+2\sum_{i=1}^{k}\sum_{j=i+1}^{k}\sum_{l=i}^{k}P_{ij,l}x_ix_j+2\sum_{i=1}^{k}\sum_{j=k+1}^{n-1}\sum_{l=i}^{k}P_{ij,l}x_ix_j\\ +2\sum_{i=1}^{k}\sum_{l=i}^{k}P_{in,l}x_i(1-x_1-x_2-\cdots-x_{n-1})\end{aligned}$$

$$\leq \sum_{i=1}^{k}\sum_{l=i}^{k} x_i^2 + 2\sum_{i=1}^{k}\sum_{j=i+1}^{k}\sum_{l=i}^{k} x_i x_j + \sum_{i=1}^{k}\sum_{j=k+1}^{n-1}\sum_{l=i}^{k} x_i x_j + \sum_{i=1}^{k}\sum_{l=i}^{k} x_i(1 - x_1 - x_2 - \cdots - x_{n-1})$$

$$= \sum_{i=1}^{k} x_i^2 + 2\sum_{i=1}^{k}\sum_{j=i+1}^{k} x_i x_j + \sum_{i=1}^{k}\sum_{j=k+1}^{n-1} x_i x_j + \sum_{i=1}^{k} x_i(1 - x_1 - x_2 - \cdots - x_{n-1})$$

$$= \sum_{i=1}^{k} x_i^2 + 2\sum_{i=1}^{k}\sum_{j=i+1}^{k} x_i x_j + \sum_{i=1}^{k}\sum_{j=k+1}^{n-1} x_i x_j + \sum_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i^2 - 2\sum_{i=1}^{k}\sum_{j=i+1}^{k} x_i x_j - \sum_{i=1}^{k}\sum_{j=k+1}^{n-1} x_i x_j$$

$$= \sum_{i=1}^{k} x_i.$$

That implies that $V(\boldsymbol{x}) \leq^b \boldsymbol{x}$. Which implies V is b-bistochastic. Putting these two parts together, this completes the theorem. □

FIXED POINT

In what follows, we will limit ourselves to three-dimensional simplex. The list of fixed points for a two-dimensional simplex had been provided by Embong and Rosli (2023). It is also noted that there are no fixed points on the interior of the two-dimensional simplex, other than the identity operator, i.e., $A_1 = D_2 = 1$ and $B_1 = B_2 = C_1 = E_2 = \frac{1}{2}$;

Theorem 6 (Embong and Rosli (2023)). *Let V be a b-bistochastic QSO defined on S^2. Then V has no interior fixed points except for the trivial case.*

The proof for this theorem is heavily dependent on the following proposition. This proposition will also be applied in our theorems.

Proposition 7. Let $0 \leq a \leq \frac{1}{2}$, $0 \leq b \leq \frac{1}{2}$ and $0 \leq c \leq 1$ and define $f(x) = \frac{1-2a-2x(b-a)}{c-2a}$. Then $f(x) \leq 0$ or $f(x) \geq 1$.

An approach similar to the paper mentioned above will be used for this paper. First, let

$$S^3 = \{x = (x, y, z, w) \in \mathbb{R}^4 \mid x, y, z, w \geq 0 \; and \; x + y + z + w \leq 1\}.$$

If $V: S^3 \to S^3$ is a b-bistochastic QSO, then V takes the form

$$V(\boldsymbol{x})_1 = f_1 = A_1x^2 + 2B_1xy + 2C_1xz + 2D_1x(1 - x - y - z),$$
$$V(\boldsymbol{x})_2 = f_2 = A_2x^2 + 2B_2xy + 2C_2xz + 2D_2x(1 - x - y - z) + E_2y^2 + 2F_2yz + 2G_2y(1 - x - y - z),$$

$$V(\boldsymbol{x})_3 = f_3 = A_3x^2 + 2B_3xy + 2C_3xz + 2D_3x(1-x-y-z) + E_3y^2 + 2F_3yz + 2G_3y(1-x-y-z) + H_3z^2 + 2I_3z(1-x-y-z).$$

Let $S_1 = \{\boldsymbol{x} \in S^3 \mid x = 0\}$, $S_2 = \{\boldsymbol{x} \in S^3 \mid y = 0\}$, $S_3 = \{\boldsymbol{x} \in S^3 \mid z = 0\}$ and $S_4 = \{\boldsymbol{x} \in S^3 \mid w = 0\}$. We also define the simplex restriction, $V|_{S_i} = \{\boldsymbol{x} \in S^3 \mid x_i = 0\}$ for $i = 1,2,3,4$. The following theorem describes the set of fixed points of b-bistochastic QSOs on a three-dimensional simplex.

Theorem 8. *Let V be a b-bistochastic QSO on S^3, then*

1. $$Fix(V|_{S_1}) = \begin{cases} \{(0,0,0,1)\} & if\, E_2 \neq 1 \text{ or } H_3 \neq 1 \\ \{(0,1,0,0),(0,0,0,1)\} & if\, E_2 = 1 \text{ and either } F_2 \neq \frac{1}{2} \text{ or } G_2 \neq \frac{1}{2} \\ \{(0,0,1,0),(0,0,0,1)\} & if\, H_3 = 1 \text{ and either } I_3 \neq \frac{1}{2} \text{ or } E_2 \neq 1 \\ \{(0,1,0,0),(0,0,1,0)\} & if\, E_2 = H_3 = 1 \text{ and } F_2 \neq \frac{1}{2} \\ \{(0,y,0,1-y)\} & if\, E_2 = 1 \text{ and } G_2 = \frac{1}{2} \\ \{(0,0,z,1-z)\} & if\, H_3 = 1 \text{ and } I_3 = \frac{1}{2} \\ \{(0,y,1-y,0)\} & if\, E_2 = H_3 = 1 \text{ and } F_2 = F_3 = \frac{1}{2} \\ \{(0,y,z,1-y-z)\} & if\, E_2 = H_3 = 1 \text{ and } F_2 = G_2 = F_3 = I_3 = \frac{1}{2} \end{cases}$$

2. $$Fix(V|_{S_2}) = \begin{cases} \{(0,0,0,1)\} & if A_1 \neq 1 \text{ or} H_3 \neq 1 \\ \{(1,0,0,0),(0,0,0,1)\} & if A_1 = 1 \text{ and either} C_1 \neq \frac{1}{2} \text{ or} D_1 \neq \frac{1}{2} \\ \{(0,0,1,0),(0,0,0,1)\} & if H_3 = 1 \text{ and either} I_3 \neq \frac{1}{2} \text{ or} A_1 \neq 1 \\ \{(1,0,0,0),(0,0,1,0)\} & if A_1 = H_3 = 1 \text{ and} C_1 \neq \frac{1}{2} \\ \{(x,0,0,1-x)\} & if A_1 = 1 \text{ and} D_1 = \frac{1}{2} \\ \{(0,0,z,1-z)\} & if H_3 = 1 \text{ and} I_3 = \frac{1}{2} \\ \{(x,0,1-x,0)\} & if A_1 = H_3 = 1 \text{ and} C_1 = C_3 = \frac{1}{2} \\ \{(x,0,z,1-x-z)\} & if A_1 = H_3 = 1 \text{ and} C_1 = D_1 = C_3 = I_3 = \frac{1}{2} \end{cases}$$

3. $$Fix(V|_{S_3}) = \begin{cases} \{(0,0,0,1)\} & if A_1 \neq 1 \text{ or} E_2 \neq 1 \\ \{(1,0,0,0),(0,0,0,1)\} & if A_1 = 1 \text{ and either} B_1 \neq \frac{1}{2} \text{ or} D_1 \neq \frac{1}{2} \\ \{(0,1,0,0),(0,0,0,1)\} & if E_2 = 1 \text{ and either} G_2 \neq \frac{1}{2} \text{ or} A_1 \neq 1 \\ \{(1,0,0,0),(0,1,0,0)\} & if A_1 = E_2 = 1 \text{ and} B_1 \neq \frac{1}{2} \\ \{(x,0,0,1-x)\} & if A_1 = 1 \text{ and} D_1 = \frac{1}{2} \\ \{(0,y,0,1-y)\} & if E_2 = 1 \text{ and} G_2 = \frac{1}{2} \\ \{(x,1-x,0,0)\} & if A_1 = E_2 = 1 \text{ and} B_1 = B_2 = \frac{1}{2} \\ \{(x,y,0,1-x-y)\} & if A_1 = E_2 = 1 \text{ and} B_1 = D_1 = B_2 = G_2 = \frac{1}{2} \end{cases}$$

4. $$Fix\left(V|_{S_4}\right) = \begin{cases} \{(0,0,1,0)\} & if A_1 \neq 1 \; or E_2 \neq 1 \\ \{(1,0,0,0),(0,0,1,0)\} & if A_1 = 1 \; and \; either B_1 \neq \frac{1}{2} \; or C_1 \neq \frac{1}{2} \\ \{(0,1,0,0),(0,0,1,0)\} & if E_2 = 1 \; and \; either F_2 \neq \frac{1}{2} \; or A_1 \neq 1 \\ \{(1,0,0,0),(0,1,0,0)\} & if A_1 = E_2 = 1 \; and B_1 \neq \frac{1}{2} \\ \{(x,0,1-x,0)\} & if A_1 = 1 \; and C_1 = \frac{1}{2} \\ \{(0,y,1-y,0)\} & if E_2 = 1 \; and F_2 = \frac{1}{2} \\ \{(x,1-x,0,0)\} & if A_1 = E_2 = 1 \; and B_1 = B_2 = \frac{1}{2} \\ \{(x,y,1-x-y,0)\} & if A_1 = E_2 = 1 \; and B_1 = C_1 = B_2 = F_2 = \frac{1}{2} \end{cases}$$

Proof. In this proof, we will consider S_1 only since the other cases for S_2, S_3 and S_4 can be done in similar manners. By Theorem 6, the b-bistochastic QSO on S_1 takes the following form:

$$f_1 = 0,$$
$$f_2 = E_2y^2 + 2F_2yz + 2G_2y(1-y-z),$$
$$f_3 = E_3y^2 + 2F_3yz + 2G_3y(1-y-z) + H_3z^2 + 2I_3z(1-y-z).$$

To solve all the fixed points, it is equivalent to solving $f_1 = x$, $f_2 = y$ and $f_3 = z$. Now, consider $f_2 = y$ that is $y = E_2y^2 + 2F_2yz + 2G_2y(1-y-z)$. Rearranging for y, one obtains

$$y\big(1 - 2G_2 - y(E_2 - 2G_2) - 2z(F_2 - G_2)\big) = 0$$
$$y = 0 \quad or \quad y = \frac{1 - 2G_2 - 2z(F_2 - G_2)}{E_2 - 2G_2}$$

By substituting $(0,0,0,1)$ into f_1, f_2 and f_3 we find that the point is a fixed point for any b-bistochastic QSOs. We will separate this proof into four cases, namely;

(1) $y = 0$,
(2) $z = 0$,
(3) $w = 0$
(4) The interior, i.e. $y \neq 0, z \neq 0$ and $w \neq 0$.

Case (1): Consider $y = 0$ and $f_3 = z$ since $f_2 = 0$ here. One obtains $f_3 = H_3z^2 + 2I_3z(1-z) = z$. By solving for z one finds

$$z\big(1 - 2I_3 - z(H_3 - 2I_3)\big) = 0,$$
$$z = 0 \quad or \quad z = \frac{1 - 2I_3}{H_3 - 2I_3}.$$

For $z = 0$ then $(0,0,0,1)$ acts as a fixed point for any cases. If $H_3 \neq 1$ then $\frac{1-2I_3}{H_3-2I_3} > 1$ which implies $(0,0,0,1)$ is the only fixed point. If $H_3 = 1$ and $I_3 \neq \frac{1}{2}$, then both $(0,0,0,1)$ and $(0,0,0,1)$ are fixed points. If $H_3 = 1$ and $I_3 = \frac{1}{2}$, then all points on the line $(0,0,z,1-z)$ are fixed points.

Case (2): Consider $z = 0$ and $f_2 = y$, then $y = 0$ or $y = \frac{1-2G_2}{E_2-2G_2}$. Using the same arguments as before, if $y = 0$ then $(0,0,0,1)$ is a fixed point for any cases. If $E_2 \neq 1$, then $\frac{1-2G_2}{E_2-2G_2} > 1$ which implies $(0,0,0,1)$ is the only fixed point. If $E_2 = 1$ but $G_2 \neq \frac{1}{2}$, then both $(0,1,0,0)$ and $(0,0,0,1)$ are fixed points. If $E_2 = 1$ and $G_2 = \frac{1}{2}$, then all points on the line $(0, y, 0, 1 - y)$ are fixed points.

Case (3): Consider $w = 0$ and $f_2 = y$, then $y = 0$ or $y = \frac{1-2F_2}{E_2-2F_2}$. If $y = 0$ implies $z = 1$, so substituting this point into f_1, f_2 and f_3 yields $f_1 = 0$, $f_2 = 0$ and $f_3 = H_3$. Therefore, if $H_3 = 1$ then the point is a fixed point. In other words, if $H_3 \neq 1$, then $(0,0,1,0)$ is not a fixed point. If $H_3 = 1$ and $E_2 \neq 1$, then $\frac{1-2F_2}{E_2-2F_2} > 1$ which $(0,0,1,0)$ is the only fixed point on this boundary. Now let $E_2 = H_3 = 1$. Under the condition $F_2 \neq \frac{1}{2}$ implies $y = 1$, hence both $(0,0,1,0)$ and $(0,1,0,0)$ are fixed points in this case. If $E_2 = 1$ and $F_2 = \frac{1}{2}$, that makes $f_2 = y$ and $f_3 = 2F_3 y(1 - y) + H_3(1 - y)^2$. In addition, if we have $F_3 = \frac{1}{2}$ and $H_3 = 1$, $f_3 = 1 - y$ in which all points on the line $(0, y, 1 - y, 0)$ are fixed points.

Case (4): On the interior of S_1, we consider only $y = \frac{1-2G_2-2z(F_2-G_2)}{E_2-2G_2}$ since $y = 0$ is not in the interior of S_1. y is akin to the form obtained by Proposition 8 since all values E_2, F_2 and G_2 here take on the same range as a, b, c. We find $y \leq 0$ or $y \geq 1$ which in both cases are not in the interior of S_1. Hence, the only fixed points are the trivial case, in which $(0, y, z, 1 - y - z)$ are all fixed points, given when $E_2 = H_3 = 1$ and $F_2 = G_2 = F_3 = I_3 = \frac{1}{2}$.

Having considered all possibilities in S_1, all fixed points are listed and this concludes the proof. □

Remark 9. Let $V: S^3 \to S^3$ be a b-bistochastic QSO. If $A_1 = E_2 = H_3 = 1$ and $B_1 = B_2 = C_1 = C_3 = D_1 = F_2 = F_3 = G_2 = I_3 = \frac{1}{2}$, the all points $\boldsymbol{x} \in S^3$ are fixed points. We will thus exclude the identity for the rest of this paper.

The following theorem proves there is no interior fixed points.

Theorem 11. *Let $V: S^3 \to S^3$ be a QSO. If V is b-bistochastic, then V has no interior fixed points, other than the trivial case.*

Proof. Let V be a b-bistochastic QSO on a three-dimensional simplex. By contradiction, assume that there exists an interior fixed point. That is $\boldsymbol{x} = (x, y, z, w)$ with $x \neq 0, y \neq 0, z \neq 0$, and $w \neq 0$. By taking the form of $V(x)_1$ from Theorem 5, one obtains

$$x = A_1 x^2 + 2B_1 xy + 2C_1 xy + 2D_1 x(1 - x - y - z).$$

Since $x \neq 0$, by dividing both sides by x one has

$$1 = A_1x + 2B_1y + 2C_1z + 2D_1(1 - x - y - z),$$

$$x = \frac{1 - 2D_1 - (2B_1 - 2D_1)y - (2C_1 - 2D_1)z}{A_1 - 2D_1}. \tag{9}$$

Then, we proceed into three cases, namely

1. $A_1 - 2D_1 = 0$,
2. $A_1 - 2D_1 < 0$,
3. $A_1 - 2D_1 > 0$.

Case (1): Let $A_1 - 2D_1 = 0$. This case only yields a potential solution if the numerator of equation (9) is 0, otherwise it will be undefined. Therefore let $1 - 2D_1 - (2B_1 - 2D_1)y - (2C_1 - 2D_1)z = 0$. However, by rearranging for y, one finds

$$y = \frac{1 - 2D_1 - 2z(C_1 - D_1)}{2B_1 - 2D_1},$$

which by Proposition 7 implies $y = 0$ or $y = 1$ which both would contradict with y being in the interior. Thus, forming a contradiction.

Case (2): Let $A_1 - 2D_1 < 0$. Since $x = \frac{1-2D_1-(2B_1-2D_1)y-(2C_1-2D_1)z}{A_1-2D_1} > 0$, that would imply

$$\begin{aligned}
&1 - 2D_1 - (2B_1 - 2D_1)y - (2C_1 - 2D_1)z < 0\\
&1 - 2B_1y - 2C_1z + 2D_1(y + z - 1) < 0\\
&2D_1(y + z - 1) < 2B_1y + 2C_1z - 1
\end{aligned}$$

Since $y + z - 1 < 0$ and $2D_1 \leq 1$, this implies $2D_1(y + z - 1) > y + z - 1$. Substituting this back into the previous inequality, one finds,

$$y + z - 1 < 2B_1y + 2C_1z - 1,$$

$$y + z < 2B_1y + 2C_1z. \tag{10}$$

However, this is impossible since $2B_1 \leq 1$ and $2C_1 \leq 1$ which implies $2B_1y + 2C_1z \leq y + z$. It is a contradiction.

Case (3): Let $A_1 - 2D_1 > 0$.

$$\begin{aligned}
x &= \frac{1 - 2D_1 - (2B_1 - 2D_1)y - (2C_1 - 2D_1)z}{A_1 - 2D_1},\\
&= \frac{1 - 2D_1}{A_1 - 2D_1}
\end{aligned}$$

$$-\frac{(2B_1-2D_1)y+(2C_1-2D_1)z}{A_1-2D_1}.$$

Adding in y and z to both sides of the equation, one obtains

$$\begin{aligned} x+y+z &= \frac{1-2D_1}{A_1-2D_1}-\left(\frac{(2B_1-2D_1)y+(2C_1-2D_1)z}{A_1-2D_1}-y-z\right), \\ &= \frac{1-2D_1}{A_1-2D_1}-\left(\frac{(2B_1-A_1)y+(2C_1-A_1)z}{A_1-2D_1}\right). \end{aligned}$$

Since $x+y+z<1$,

$$\begin{aligned} \frac{1-2D_1}{A_1-2D_1}-\left(\frac{(2B_1-A_1)y+(2C_1-A_1)z}{A_1-2D_1}\right) &< 1, \\ 1-2D_1-(2B_1-A_1)y-(2C_1-A_1)z &< A_1-2D_1, \\ 1-A_1 &< 2B_1y+2C_1z-A_1y-A_1z, \\ A_1(y+z-1) &< 2B_1y+2C_1z-1. \end{aligned}$$

By following the same argument in case (2), one would arrive at the same contradiction with (10). Therefore, there are no fixed point in the interior of the simplex. This proves the theorem. □

This theorem is rather significant as it implies that all the limiting points must be on the boundary of the simplex.

DYNAMICS

One significant theorem that had been proven previously is the limiting point must exist for any b-bistochastic QSOs. The theorem is given below.

Theorem 11(Mukhamedov and Embong (2015)). *Let V be a b-bistochastic QSO defined on S^{n-1}, then the limit $\lim_{m\to\infty} V^{(m)}(\boldsymbol{x})$ exists any $\boldsymbol{x}\in S^{n-1}$.*

Another interesting theorem on the dynamics of b-bistochastic QSOs is that the point $(0,0,\dots,0,1)$ is always an attracting fixed point under specific conditions.

Theorem 12(Mukhamedov and Embong (2015)). *Let V be a b-bistochastic QSO. If $P_{lj,l}<\frac{1}{2}$ for $l=1,2,\dots,n-1$ then $\boldsymbol{x}=(0,0,\dots,0,1)$ is an attracting fixed point.*

We provide the following example for the illustration of Theorem 12.

Example 13. Consider the b-bistochastic QSO.

$$\begin{aligned} V(x)_1 &= 0.2x^2+0.6xy+0.4xz+0.3x(1-x-y-z) \\ V(x)_2 &= 0.4x^2+0.8xy+0.2xz+0.1x(1-x-y-z)+0.4y^2+0.6yz \\ &\quad +0.5y(1-x-y-z) \end{aligned}$$

$$V(x)_3 = 0.1x^2 + 0.2xy + 0.6xz + 0.4x(1-x-y-z) + 0.6yz + 0.3y(1-x-y-z) + 0.5z^2 + 0.2z(1-x-y-z)$$

By Theorem 8, one of the fixed point of this system is $(0,0,0,1)$. Choose any initial point and iterate the point, we can see that the trajectory approaches $(0,0,0,1)$. This agrees with Theorem 13. The following table to show several initial points and their trajectory 500 iterations.

Table 1: Trajectory of some initial points for Example 13

Initial Point	Trajectory after 500 iterations
(0.1,0.2,0.3,0.4)	(0,0,0,1)
(1,0,0,0)	(0,0,0,1)
(0.3,0.5,0.1,0.1)	(0,0,0,1)
(0.5,0.4,0.1,0)	(0,0,0,1)
(0.35,0.15,0.05,0.45)	(0,0,0,1)

As for the following example, we consider the converse of Theorem 12, that is if $P_{lj,l} = \frac{1}{2}$.

Example 14. Consider the following b-bistochastic QSO.

$$V(x)_1 = 0.5x^2 + 0.4xy + 0.6xz + 0.2x(1-x-y-z)$$
$$V(x)_2 = 0.2x^2 + xy + 0.1xz + 0.8x(1-x-y-z) + y^2 + 0.4yz + y(1-x-y-z)$$
$$V(x)_3 = 0.1x^2 + 0.8xz + 0.6yz + 0.6z^2 + 0.4z(1-x-y-z)$$

The Jacobian is given by,

$$J = \begin{bmatrix} x + 0.4y + 0.6z + 0.2w & 0.4x & 0.6x & 0.2x \\ 0.4x + y + 0.1z + 0.8w & x + 2y + 0.4z + w & 0.1x + 0.4y & 0.8x + y \\ 0.2x + 0.8z & 0.6z & 0.8x + 0.6y + 1.2z + 0.4w & 0.4z \\ 0.4x + 0.6y + 0.5z + w & 0.6x + z + w & 0.5x + y + 0.8z + 1.6w & x + y + 1.6z + 2w \end{bmatrix}$$

By Theorem 8, the fixed points of this QSO must be $(0, y, 0, 1-y)$. By substituting this fixed point into the Jacobian function, one would find that the eigenvalue to be $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0.2 + 0.2y$ and $\lambda_4 = 0.4 + 0.2y$. Therefore, this fixed point is non-hyperbolic.

Similar to Example 13, choose several initial points and their trajectory after 500 iterations.

Table 2: Trajectory of some initial points for Example 14

Initial Point	Trajectory after 500 iterations
(0.7,0.05,0.2,0.05)	(0,0.113407,0,0.886593)
(0.1,0.2,0.3,0.4)	(0,0.14414,0,0.85586)
(0.2,0.25,0.4,0.15)	(0,0.149684,0,0.850316)
(0.10,0.45,0.15,0.30)	(0,0.353462,0,0.646538)
(0.2,0.15,0.6,0.05)	(0,0.803162,0,0.196838)

Notice that the initial points eventually converge to a point on the line $(0, y, 0, 1 - y)$. However, the limiting point is dependent on the initial point. This shows the needs of a more detailed study to fully comprehend the omega-limiting set.

CONCLUSION

In this paper, we determined the descriptions of b-bistochastic QSOs on any finite dimensional simplex. The fixed points of b-bistochastic QSOs had also been found as well as some examples of b-bistochastic QSOs with their descriptions. Another refinement on previous theorems is we had also improved a theorem fixed points on the interior of a simplex up to three-dimensional simplex.

Some future works that may be suggested are to find all the limiting behaviour of a b-bistochastic QSO on a three-dimensional simplex and any finite-dimensional simplexes.

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