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Toeplitz Determinant for a Sublcass of Tilted Starlike Functions Associated with Epicycloid Domain

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ABSTRACT

This study investigates a subclass of analytic and univalent functions defined in the open unit disk. A new class of tilted starlike functions associated with the epicycloid domain is introduced. The defining condition for this class is expressed through a subordination relation involving a rotation parameter and the epicycloid mapping. Within this framework, we establish coefficient bounds for functions belonging to the class. In addition, we examine the Toeplitz determinants which provide further structural insights into the behavior of these functions. The results contribute to the broader understanding of analytic function theory and highlight new connections within Geometric Function Theory (GFT).

Keywords: Tilted starlike functions, Epicycloid domain, Coefficient bounds, Toeplitz determinants

INTRODUCTION

We let $H(E)$ be denoted as the family of holomorphic (analytic) functions f in an open unit disk $E = \{\mathfrak{I} : \mathfrak{I} \in £ \text{ and } |\mathfrak{I}| < 1\}$. Consider the class $\Lambda \subset H(E)$, consisting of function f which are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ can be expressed in the form of the expansion in terms of a Taylor series

$$f(\mathfrak{I}) = \mathfrak{I} + \sum_{n=2}^{\infty} a_n \mathfrak{I}^n, \ \mathfrak{I} \in E. \tag{1}$$

The symbol S represents the subclass of which consists of univalent functions that are one-to-one within the unit disk E. Among the well-studied subclasses of S include the classical and widely explored classes of starlike and convex functions in GFT (Duren, 1983).

As established by Miller and Mocanu (1981), given two functions $f, d \in H(E)$, the function f is subordinate to d, denoted as $f \prec d$, provided that an analytic function ϖ (a Schwarz function) exists that satisfy $\varpi(0)=0$ and $|\varpi(\Im)|<1$ such that $f(\Im)=d(\varpi(\Im))$ for all $\Im \in E$. In fact, if $d \in S$, then $f \prec d$ necessarily holds that $f(0)=d(0)$ and $f(E) \subset d(E)$.

Additionally, Ma and Minda (1994) introduced a general framework for defining subclasses of starlike functions through the principle of subordination, offering a general formulation widely used in GFT, defined as

$$S^*(\psi) := \left\{ f \in \Lambda : \frac{\Im f'(\Im)}{f(\Im)} \prec \psi(\Im) \right\}, \tag{2}$$

where the function ψ is analytic in E and having a real part greater than zero, mapping E onto a starlike domain centered at 1 and symmetric about the real axis. Furthermore, ψ is normalized so that $\psi(0)=1$ and $\psi'(0)>0$.

In recent years, numerous subclasses of $S^*(\psi)$ have been studied by choosing specific forms of the function ψ. For instance, Sharma et al. (2016) introduced the class $S^*_{\text{car}}(\psi) = 1 + \frac{4}{3}\Im + \frac{2}{3}\Im^2$, which maps onto a cardioid-shaped region. Similarly, Gupta et al. (2021) proposed another cardioid-related subclass defined by $S^*_{\text{car}} = S^*(\psi)$, where $\psi(\Im) = 1 + \Im + \frac{1}{2}\Im^2$. Wani and Swaminathan (2020) introduced a subclass related to nephroid domain as $\psi_{Ne}(z) = 1 + \Im - \frac{1}{3}\Im^3$. Several authors have studied subclasses of starlike functions linked to epicycloid domains. By definition, an epicycloid is a curve in the plane formed by the circumference of a fixed point on the edge of a radius circle b as it rotates externally along a fixed radius circle a, without slipping (Gandhi et al., 2022). Gandhi (2020) introduced and defined a subclass of starlike functions associated with three-leaf epicycloid as follows:

$$S^*_{3,L} := \left\{ f \in \Lambda : \frac{\Im f'(\Im)}{f(\Im)} \prec 1 + \frac{4}{5}\Im + \frac{1}{5}\Im^4 \right\}, \ \Im \in E. \tag{3}$$

Subsequently, Gandhi et al. (2022) considered a more general form of the function associated with an epicycloid having $n-1$ cusps for $n \geq 2$, which adheres to the framework proposed by Ma and Minda (1994) for admissibility in the unified class of starlike functions. The open unit disk E is mapped by this function onto a region enclosed by an epicycloid that has cusps. The corresponding subclass of starlike functions is defined as

$$S^*_{n-1,L} := \left\{ f \in \Lambda : \frac{\Im f'(\Im)}{f(\Im)} \prec 1 + \frac{n}{n+1}\Im + \frac{1}{n+1}\Im^n \right\}, \ \Im \in E. \tag{4}$$

In a recent contribution, Shi et al. (2022) studied the subclass of $S^*_{4,L}$ as follows:

$$S_{4,L}^{*} := \left\{ f \in \Lambda : \frac{\Im f'(\Im)}{f(\Im)} \prec 1 + \frac{5}{6}\Im + \frac{1}{6}\Im^{5} \right\},\ \Im \in E. \tag{5}$$

Exploring starlike subclasses is fundamental in univalent function theory, as it facilitates broader generalizations of classical types and enhances understanding of their functional properties. An important development in this direction is the introduction of tilted factor, which provides a broader and more flexible view of starlike behavior. The foundation of the study on the function $e^{i\alpha}$, $|\alpha| \le \pi$ was further explored by Mohamad (2000), who proposed a generalization incorporating a rotational parameter, defined for $|\alpha| \le \pi$, $0 \le \delta < 1$ and $t_{\alpha\delta} = \cos\alpha - \delta > 0$. The functions from this class can be represented as

$$p(\Im) = \frac{e^{i\alpha} f'(\Im) - i\sin\alpha - \delta}{t_{\alpha\delta}}. \tag{6}$$

Building upon the foundational work of Halim (1991), Dahhar and Janteng (2009), and El-Ashwah and Thomas (1987), Wahid et al. (2015) proposed a further extension in the form of tilted starlike functions characterized in relation to conjugate points, denoted by $S_{C}^{*}(\varepsilon,\delta,A,B)$ and fulfilling the condition

$$\left\{ e^{i\varepsilon} \frac{\Im f'(\Im)}{f(\Im)} - \delta - i\sin\varepsilon \right\} \frac{1}{t_{\varepsilon\delta}} \prec \frac{1 + A\Im}{1 + B\Im},\ -1 \le B < A \le 1. \tag{7}$$

Motivated by the general class $S_{n-1,L}^{*}$ introduced by Gandhi et al. (2022), the analytic estimates established in Shi et al. (2022), and the tilted component incorporated by Wahid et al. (2015), the present work aims to generalize $S_{n-1,L}^{*}$ by introducing a tilted factor. This leads us to define the following generalized subclass:

Definition 1.1. *A function belongs to the class* $S_{T,n-1}^{*}(\varepsilon)$ *if and only if*

$$\left\{ e^{i\varepsilon} \frac{\Im f'(\Im)}{f(\Im)} - i\sin\varepsilon \right\} \frac{1}{\cos\varepsilon} \prec 1 + \frac{n}{n+1}\Im + \frac{1}{n+1}\Im^{n},\ \Im \in E. \tag{8}$$

where $n \ge 2$ and $|\varepsilon| < \frac{\pi}{2}$. From the principle of subordination, we have that $f \in S_{T,n-1}^{*}(\varepsilon)$ in the case where

$$\left\{ e^{i\varepsilon} \frac{\Im f'(\Im)}{f(\Im)} - i\sin\varepsilon \right\} \frac{1}{\cos\varepsilon} = 1 + \frac{n}{n+1}\varpi(\Im) + \frac{1}{n+1}\left[\varpi(\Im)\right]^{n},\ \varpi \in H(E). \tag{9}$$

The analytic properties of this subclass are investigated through coefficient-based functionals, with particular emphasis on the Toeplitz determinant. Toeplitz matrices are among the well-known classes of structured matrices and share a close connection with Hankel matrices. While Hankel matrices are characterized by constant values along their anti-diagonals, Toeplitz matrices are defined by having constant elements along each diagonal (Ali et al., 2018). Although the symmetric Toeplitz determinant has been discussed in the earlier literature, some of the initial sources are no longer considered reliable. In the present work, we adopt the formulation consistent with later studies, such as that of Ali et al. (2018), where the Toeplitz determinant is utilized in the context of the theory of analytic functions.

Assume a function has the form described in (1), the symmetric structure associated with the Toeplitz determinant, $\mathrm{T}_{q,n}$, with $a_1 = 1$ and $n, q \geq 1$ is given by

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}.$$

Hence, $\mathrm{T}_{q,n}$ takes the following form for the specific choices of q and n :

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix} = a_2^2 - a_3^2,$$

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix} = 1\left(1 - a_2^2\right) - a_2\left(a_2 - a_2 a_3\right) + a_3\left(a_2^2 - a_3\right),$$

$$T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_2 & a_2 \end{vmatrix} = a_2\left(a_2^2 - a_2 a_3\right) - a_3\left(a_2 a_3 - a_3 a_4\right) + a_4\left(a_2 a_3 - a_2 a_4\right),$$

and so on.

Several studies have focused on estimating the Toeplitz determinants within various subclasses of analytic functions. For instance, Ali et al. (2018) established sharp bounds for the Toeplitz determinants $T_2(n)$, $T_3(1)$, and $T_3(2)$ for functions in the subclass of starlike functions. In the context of starlike functions related to the sine function, Zhang et al. (2019) examined $T_3(2)$; see also Zhang and Tang (2021), who extended this to derive the upper bound for $T_4(2)$. Yunus et al. (2020) obtained a sharp estimate for $T_2(2)$ belonging to the class of functions related to the generalized cardioid domain. More recently, refer to Wahid and Mohamad (2021) for the study of determinants $T_2(2)$, $T_3(1)$, and $T_3(2)$ under the class $S_C^*(\varepsilon, \delta, A, B)$, which generalized several known results in the existing literature.

This paper presents the derivation up to fourth coefficient bounds along with the Toeplitz determinants $T_2(2)$, $T_3(1)$, and $T_3(2)$ for functions within the class $S_{T,n-1}^*(\varepsilon)$.

PRELIMINARY RESULTS

Consider be denoted as the class of Carathéodory functions, which consists of analytic functions given by

$$p(\Im)=1+\sum_{n=1}^{\infty}c_n\Im^n, \tag{10}$$

where $\mathrm{Re}(p(\Im))>0$ $(\Im\in E)$. The main results are substantiated by the lemmas provided in this section.

Lemma 2.1. (Duren (1983)). *Assuming that* $p\in P$ *is described by (10), then* $|c_n|\leq 2$ *holds for every* $n\geq 1$.

Lemma 2.2. (Efraimidis (2016)). *If* $p\in P$ *is described by (10) with* $\mu\in£$. *Then*

$$|c_m-\mu c_k c_{m-k}|\leq 2\max\{1,|2\mu-1|\},\text{for}\,1\leq k\leq m-1. \tag{11}$$

MAIN RESULTS

We first derive the coefficient bounds up to the fourth order for functions of the class $S^*_{T,n-1}(\varepsilon)$.

Theorem 3.1. *If* $f\in S^*_{T,n-1}(\varepsilon)$ *be the expression given in (1), then*

$$|a_2|\leq\frac{n\cos\varepsilon}{(n+1)}, \tag{12}$$

$$|a_3|\leq\frac{n\cos\varepsilon}{2(n+1)}, \tag{13}$$

and

$$|a_4|\leq\frac{n\cos\varepsilon}{3(n+1)}\left[\left|1-\frac{3ne^{-i\varepsilon}\cos\varepsilon}{2(n+1)}+\frac{n^2e^{-i2\varepsilon}\cos^2\varepsilon}{2(n+1)^2}\right|+1\right]. \tag{14}$$

Proof. Assume that $f\in S^*_{T,n-1}(\varepsilon)$. Then, Then, by utilizing the subordination theory, it follows that a Schwarz function exists, for which $\varpi(0)=0$ and $|\varpi(\Im)|<1$, leading to

$$\left\{e^{i\varepsilon}\frac{\Im f'(\Im)}{f(\Im)}-i\sin\varepsilon\right\}\frac{1}{\cos\varepsilon}=1+\frac{n}{n+1}\varpi(\Im)+\frac{1}{n+1}\left[\varpi(\Im)\right]^n=\gamma(\Im) \tag{15}$$

Let the function be constructed as

$$p\left(\Im\right)=\frac{1+\varpi\left(\Im\right)}{1-\varpi\left(\Im\right)}=1+c_1\Im+c_2\Im^2+c_3\Im^3+\cdots. \tag{16}$$

It is evident that $p\in P$. It follows that

$$\begin{aligned}\varpi(\Im)&=\frac{p(\Im)-1}{p(\Im)+1}\\&=\frac{1}{2}c_1\Im+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)\Im^2+\left(\frac{1}{8}c_1^3-\frac{1}{2}c_1c_2+\frac{1}{2}c_3\right)\Im^3\\&\quad+\left(\frac{1}{2}c_4-\frac{1}{2}c_1c_3-\frac{1}{4}c_2^2-\frac{1}{16}c_1^4+\frac{3}{8}c_1^2c_2\right)\Im^4+\cdots.\end{aligned} \tag{17}$$

Now, from (1), we obtain

$$\begin{aligned}\frac{\Im f'(\Im)}{f(\Im)}&=1+a_2\Im+\left(2a_3-a_2^2\right)\Im^2+\left(a_2^3-3a_2a_3+3a_4\right)\Im^3\\&\quad+\left(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2\right)\Im^4+\cdots.\end{aligned} \tag{18}$$

We now rewrite the left side of equation (15) in the following form

$$\begin{aligned}\left(e^{i\varepsilon}\frac{\Im f'(\Im)}{f(\Im)}-i\sin\varepsilon\right)\frac{1}{\cos\varepsilon}&=1+\frac{e^{i\varepsilon}a_2}{\cos\varepsilon}\Im+\frac{e^{i\varepsilon}(2a_3-a_2^2)}{\cos\varepsilon}\Im^2+\frac{e^{i\varepsilon}(a_2^3-3a_2a_3+3a_4)}{\cos\varepsilon}\Im^3\\&\quad+\frac{e^{i\varepsilon}(4a_5-a_2^4+4a_2^2a_3-4a_2a_4-2a_3^2)}{\cos\varepsilon}\Im^4+\cdots.\end{aligned} \tag{19}$$

By inserting the expansion given in (17) into equation (15), the right side of the expression (15) is simplified to

$$\begin{aligned}\gamma(\Im)&=1+\frac{n}{2(n+1)}c_1\Im+\left(\frac{n}{2(n+1)}c_2-\frac{n}{4(n+1)}c_1^2\right)\Im^2\\&\quad+\left(\frac{n}{2(n+1)}c_3-\frac{n}{2(n+1)}c_1c_2+\frac{n}{8(n+1)}c_1^3\right)\Im^3.\end{aligned} \tag{20}$$

By equating the expressions in (19) and (20), yield

$$a_2=\frac{n\cos\varepsilon}{2\left(n+1\right)e^{i\varepsilon}}c_1, \tag{21}$$

$$a_3=\frac{n\cos\varepsilon}{8\left(n+1\right)^2e^{i\varepsilon}}\left[2\left(n+1\right)c_2-\left(\frac{\left(n+1\right)e^{i\varepsilon}-n\cos\varepsilon}{e^{i\varepsilon}}\right)c_1^2\right], \tag{22}$$

and

$$a_4 = \frac{n\cos\varepsilon}{48(n+1)^3 e^{i\varepsilon}} \begin{bmatrix} \left(2(n+1)^2 - 2n^2 \frac{\cos^2\varepsilon}{e^{i2\varepsilon}} - 3n\frac{\cos\varepsilon}{e^{i\varepsilon}} \left(\frac{(n+1)e^{i\varepsilon} - n\cos\varepsilon}{e^{i\varepsilon}} \right) \right) c_1^3 \\ -(n+1)\left(\frac{8(n+1)e^{i\varepsilon} - 6n\cos\varepsilon}{e^{i\varepsilon}} \right) c_1 c_2 + 8(n+1)^2 c_3 \end{bmatrix}. \tag{23}$$

By applying **Lemma 2.1**, we obtain

$$|a_2| \leq \frac{n\cos\varepsilon}{n+1}.$$

Utilizing **Lemma 2.2** with $\mu = \frac{1}{2} - \frac{ne^{-i\varepsilon}\cos\varepsilon}{2(n+1)}$, we get

$$|a_3| \leq \frac{n}{2(n+1)} \left| \frac{\cos\varepsilon}{e^{i\varepsilon}} \right| \max\left\{ 1, \left| \frac{n\cos\varepsilon}{(n+1)e^{i\varepsilon}} \right| \right\}.$$

Since $|e^{i\varepsilon}| = 1$, the bound of $|a_3|$ is

$$|a_3| \leq \frac{n\cos\varepsilon}{2(n+1)}.$$

To estimate the bound of $|a_4|$, we consider the expression

$$|a_4| \leq \left| \frac{ne^{-i\varepsilon}\cos\varepsilon}{6(n+1)} \left\{ \left(\frac{1}{4} + \frac{n^2 e^{-i2\varepsilon}\cos^2\varepsilon}{8(n+1)^2} - \frac{3ne^{-i\varepsilon}\cos\varepsilon}{8(n+1)} \right) c_1^3 + \left[c_3 - \left(1 - \frac{3ne^{-i\varepsilon}\cos\varepsilon}{4(n+1)} \right) c_1 c_2 \right] \right\} \right|.$$

Through the use of **Lemma 2.1** and **Lemma 2.2**, along with $|e^{-i\varepsilon}| = 1$, we get

$$|a_4| \leq \frac{n\cos\varepsilon}{3(n+1)} \left[\left| 1 - \frac{3ne^{-i\varepsilon}\cos\varepsilon}{2(n+1)} + \frac{n^2 e^{-i2\varepsilon}\cos^2\varepsilon}{2(n+1)^2} \right| + 1 \right].$$

This completes the proof of **Theorem 3.1.**

Theorem 3.2. *If* $f \in S^*_{T,n-1}(\varepsilon)$ *be the expression given in (1), then*

$$|T_2(2)| \leq \frac{5n^2\cos^2\varepsilon}{4(n+1)^2}. \tag{24}$$

Proof. If $f \in S^*_{T,n-1}(\varepsilon)$ of the form (1), then clearly

$$\left|T_2(2)\right| = \left|a_2^2 - a_3^2\right|.$$

By applying triangle inequality, we get

$$\left|T_2(2)\right| \le \left|a_2^2\right| + \left|a_3^2\right|. \tag{25}$$

Hence, from (12) and (13), squaring $|a_2|$ and $|a_3|$, we get

$$\left|a_2^2\right| \le \frac{n^2\cos^2\varepsilon}{(n+1)^2}, \tag{26}$$

and

$$\left|a_3^2\right| \le \frac{n^2\cos^2\varepsilon}{4(n+1)^2}. \tag{27}$$

By substituting (27) and (26) into (25), we obtain the desired result

$$\left|T_2(2)\right| \le \frac{5n^2\cos^2\varepsilon}{4(n+1)^2}.$$

This completes the proof of **Theorem 3.2.**

Theorem 3.3. *If $f \in S^*_{T,n-1}(\varepsilon)$ be the expression given in (1), then*

$$\left|T_3(1)\right| \le 1 + \frac{n^2\cos^2\varepsilon}{(n+1)^2}\left[2 + \frac{1}{4}\max\left\{1, \left|\frac{3n\cos\varepsilon}{(n+1)}\right|\right\}\right]. \tag{28}$$

Proof. If $f \in S^*_{T,n-1}(\varepsilon)$, then by expanding the determinant $T_3(1)$, we get

$$\left|T_3(1)\right| \le 1 + 2\left|a_2^2\right| + |a_3|\left|a_3 - 2a_2^2\right|. \tag{29}$$

Equations (21) and (22) together yield

$$\begin{aligned}
\left|a_3 - 2a_2^2\right| &= \left|\frac{n\cos\varepsilon}{8(n+1)^2 e^{i\varepsilon}}\left[2(n+1)c_2 - \left((n+1) - \frac{n\cos\varepsilon}{e^{i\varepsilon}}\right)c_1^2\right] - 2\left(\frac{n\cos\varepsilon}{2(n+1)e^{i\varepsilon}}c_1\right)^2\right| \\
&= \left|\frac{n\cos\varepsilon}{4(n+1)e^{i\varepsilon}}\left[c_2 - \left(\frac{1}{2(n+1)}\left((n+1) - \frac{n\cos\varepsilon}{e^{i\varepsilon}}\right) + \frac{2n\cos\varepsilon}{(n+1)e^{i\varepsilon}}\right)c_1^2\right]\right| \\
&= \left|\frac{n\cos\varepsilon}{4(n+1)e^{i\varepsilon}}\right|\left|c_2 - \left(\frac{(n+1)e^{i\varepsilon} + 3n\cos\varepsilon}{2(n+1)e^{i\varepsilon}}\right)c_1^2\right|.
\end{aligned}$$

By using **Lemma 2.2**, with $\mu = \dfrac{(n+1)e^{i\varepsilon} + 3n\cos\varepsilon}{2(n+1)e^{i\varepsilon}}$, we compute

$$|a_3 - 2a_2^2| \le \left|\frac{n\cos\varepsilon}{2(n+1)e^{i\varepsilon}}\right| \max\left\{1, \left|\frac{3n\cos\varepsilon}{(n+1)e^{i\varepsilon}}\right|\right\}.$$

Since $|e^{i\varepsilon}| = 1$, then

$$|a_3 - 2a_2^2| \le \frac{n\cos\varepsilon}{2(n+1)} \max\left\{1, \frac{3n\cos\varepsilon}{n+1}\right\}. \tag{30}$$

Now, by substituting the values from (13), (26) and (30) into (29), we obtain

$$|T_3(1)| \le 1 + 2\left(\frac{n^2\cos^2\varepsilon}{(n+1)^2}\right) + \frac{n\cos\varepsilon}{2(n+1)} \cdot \frac{n\cos\varepsilon}{2(n+1)} \max\left\{1, \frac{3n\cos\varepsilon}{n+1}\right\}.$$

By simplifying the inequality, we get the desired result

$$|T_3(1)| \le 1 + \frac{n^2\cos^2\varepsilon}{(n+1)^2}\left[2 + \frac{1}{4}\max\left\{1, \left|\frac{3n\cos\varepsilon}{(n+1)}\right|\right\}\right].$$

This completes the proof of **Theorem 3.3.**

Theorem 3.4. *If* $f \in S^*_{T,n-1}(\varepsilon)$ *be the expression given in (1), then*

$$|T_3(2)| \le \frac{3n^3\cos^3\varepsilon}{2(n+1)^3}\left(\frac{1}{3}\beta + \frac{4}{3}\right), \tag{31}$$

where $\beta = \left|1 - \dfrac{3ne^{-i\varepsilon}\cos\varepsilon}{2(n+1)} + \dfrac{n^2e^{-i2\varepsilon}\cos^2\varepsilon}{2(n+1)^2}\right|$.

Proof. If $f \in S^*_{T,n-1}(\varepsilon)$ be the expression given in (1), then the determinant of $T_3(2)$ can be written as

$$|T_3(2)| \le |a_2 - a_4|\left(|a_2^2 - a_3^2| + |a_2a_4 - a_3^2|\right). \tag{32}$$

By the application of triangle inequality and substitutions of the values from (12) and (14), we get

$$\begin{aligned}|a_2-a_4| &\le |a_2|+|a_4| \\ &\le \frac{n\cos\varepsilon}{n+1}+\frac{n\cos\varepsilon}{3(n+1)}\left[\left|1-\frac{3ne^{-i\varepsilon}\cos\varepsilon}{2(n+1)}+\frac{n^2e^{-2i\varepsilon}\cos^2\varepsilon}{2(n+1)^2}\right|+1\right] \\ &= \frac{n\cos\varepsilon}{n+1}\left(1+\frac{1}{3}\left[\left|1-\frac{3ne^{-i\varepsilon}\cos\varepsilon}{2(n+1)}+\frac{n^2e^{-2i\varepsilon}\cos^2\varepsilon}{2(n+1)^2}\right|+1\right]\right).\end{aligned} \tag{33}$$

Next, by applying second-order Hankel determinant associated with functions in S, we obtain

$$|a_2a_4-a_3^2| \le \frac{n^2\cos^2\varepsilon}{4(n+1)^2}. \tag{34}$$

Upon substitution of the values from (24), (33) and (34) into (32), we obtain the result

$$\begin{aligned}|T_3(2)| &\le \frac{n\cos\varepsilon}{(n+1)}\left(1+\frac{1}{3}\left[\left|1-\frac{3ne^{-i\varepsilon}\cos\varepsilon}{2(n+1)}+\frac{n^2e^{-i2\varepsilon}\cos^2\varepsilon}{2(n+1)^2}\right|+1\right]\right)\left(\frac{5n^2\cos^2\varepsilon}{4(n+1)^2}+\frac{n^2\cos^2\varepsilon}{4(n+1)^2}\right) \\ &= \frac{3n^3\cos^3\varepsilon}{2(n+1)^3}\left(\frac{1}{3}\beta+\frac{4}{3}\right),\end{aligned}$$

where $\beta=\left|1-\frac{3ne^{-i\varepsilon}\cos\varepsilon}{2(n+1)}+\frac{n^2e^{-i2\varepsilon}\cos^2\varepsilon}{2(n+1)^2}\right|$.

This completes the proof of **Theorem 3.4.**

By assigning specific values to the parameters in **Theorems 3.1–3.4**, we derive the following corollaries:

Corollary 3.1. *If* $f\in S^*_{T,n-1}(0)$, we get

$$|a_2| \le \frac{n}{(n+1)}, \tag{35}$$

$$|a_3| \le \frac{n}{2(n+1)}, \tag{36}$$

and

$$|a_4| \le \frac{n}{3(n+1)}\left[\left|1-\frac{3n}{2(n+1)}+\frac{n^2}{2(n+1)^2}\right|+1\right]. \tag{37}$$

The inequality in (35) and (36) of ***Corollary 3.1*** *aligns to the sharp coefficient bounds obtained by Gandhi et al. (2022) for the class* $S^*_{n-1,L}$.

Corollary 3.2. *If* $f \in S^*_{T,n-1}(0)$, we get

$$|T_2(2)| \le \frac{5n^2}{4(n+1)^2},$$

$$|T_3(1)| \le 1 + \frac{n^2}{(n+1)^2}\left[2 + \frac{1}{4}\max\left\{1, \frac{3n}{(n+1)}\right\}\right],$$

and

$$|T_3(2)| \le \frac{3n^3}{2(n+1)^3}\left(\frac{1}{3}\beta + \frac{4}{3}\right),$$

where $\beta = \left|1 - \frac{3n}{2(n+1)} + \frac{n^2}{2(n+1)^2}\right|$.

Corollary 3.3. *If* $f \in S^*_{T,3}(0)$, we get

$$|T_2(2)| \le \frac{4}{5},$$

$$|T_3(1)| \le 2.664,$$

and

$$|T_3(2)| \le 1.0547.$$

Corollary 3.4. *If* $f \in S^*_{T,4}(0)$, we get

$$|T_2(2)| \le \frac{125}{144},$$

$$|T_3(1)| \le 2.8229,$$

and

$$|T_3(2)| \le 1.1855.$$

CONCLUSION

This article investigates coefficient bounds and Toeplitz determinants for a subclass of tilted starlike functions associated with the epicycloid domain. Through rigorous analysis, we establish bounds up to the third coefficient, which reduce to the sharp results obtained by Gandhi et al.

(2022), as highlighted in **Corollary 3.1**. It is observed that the coefficient bounds for the starlike class, $S^{*}_{n-1,L}$ and the tilted starlike class, $S^{*}_{T,n-1}(\varepsilon)$ coincide. Furthermore, by substituting specific parameter values into the defining condition of the subclass $S^{*}_{T,n-1}(\varepsilon)$, we derive explicit estimates for the Toeplitz determinants $T_2(2)$, $T_3(1)$, and $T_3(2)$ presented in **Corollary 3.2-3.4**. The findings mark the first investigation of Toeplitz determinants within the context of the epicycloid domain. While the subclass itself has been studied in earlier works, the Toeplitz determinant results do not reduce to any existing subclasses. This highlights the novelty of the contribution and raises an important open problem of whether sharp bounds can be established in future studies. Overall, this study enhances the understanding of univalent functions by examining the relationship between their coefficient growth and associated Toeplitz determinants. The findings extend existing knowledge in GFT and may serve as a foundation for future research in complex analysis and related areas.

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